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On a result of G. Bennett

by
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Abstract

The aim of this paper is to extend a result due to G. Bennett to the framework of signed Borel measures and also to higher dimensions. Our extension is an illustration of the phenomenon of concentration convexity at endpoints.

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In a recent paper, G. Bennett [1] noticed a number of interesting consequences of an inequality describing the behavior of convex functions with respect to a mass distribution. An abstract version of his result, outlining the phenomenon of *concentration convexity to endpoints*, is as follows:

Theorem 1. *Suppose that I is an interval carrying a positive Borel measure μ and A, B, C are three nonoverlapping compact subintervals of I of positive measure.*

Then

(i) $\mu(B) = \mu(A) + \mu(C)$; *and*

(ii) $\int_B t d\mu(t) = \int_A t d\mu(t) + \int_C t d\mu(t)$

provide a set of necessary and sufficient conditions under which every convex function f defined on I verifies the inequality

$$\int_B f(t)d\mu(t) \leq \int_A f(t)d\mu(t) + \int_C f(t)d\mu(t).$$

The Necessity part follows from the case of affine functions (that is, of the form $\alpha + \beta t$). The Sufficiency part is based on the following identity,

$$\frac{1}{\mu(B)} \int_B t d\mu(t) = \frac{\mu(A)}{\mu(B)} \left(\frac{1}{\mu(A)} \int_A t d\mu(t) \right) + \frac{\mu(C)}{\mu(B)} \left(\frac{1}{\mu(C)} \int_C t d\mu(t) \right),$$

which tells us that the barycenter

$$b(\mu|B) = \frac{1}{\mu(B)} \int_B t d\mu(t),$$

of the restriction $\mu|B$, of μ to B, is a convex combination of the barycenters of the restrictions of μ to A and to C. Thus B lies between A and C. Let $L = \alpha + \beta t$ be the affine function joining the endpoints of B. Since f is a convex function, the following inequalities occur:

$$f|_A \geq L, \; f|_C \geq L \text{ and } L \geq f|_B. \tag{1}$$

Therefore

$$\begin{aligned}
\int_B f(t)d\mu(t) &\leq \int_B L(t)d\mu(t) = \alpha\mu(B) + \beta \int_B td\mu(t) \\
&= \alpha\mu(A) + \beta \int_A td\mu(t) + \alpha\mu(C) + \beta \int_C td\mu(t) \\
&= \int_A L(t)d\mu(t) + \int_C L(t)d\mu(t) \\
&\leq \int_A f(t)d\mu(t) + \int_C f(t)d\mu(t),
\end{aligned}$$

and the proof of Theorem 1 is complete.

Interestingly, Theorem 1 works outside the framework of positive measures. In fact, the possibility to integrate inequalities of the form (1) can be ascribed also to certain signed Borel measures (of positive total mass).

Definition 1. A *Steffensen-Popoviciu measure* is any real Borel measure μ on an interval I such that $\mu(I) > 0$ and

$$\int_I f(x)\, d\mu(x) \geq 0 \quad \text{for every nonnegative convex function} \quad f : I \to \mathbb{R}.$$

A complete characterization of this concept is offered by the following result, independently due to T. Popoviciu [5] and A. M. Fink [2]:

Lemma 1. Let μ be a real Borel measure on an interval I with $\mu(I) > 0$. Then μ is a Steffensen-Popoviciu measure if, and only if, it verifies the following condition of endpoints positivity,

$$\int_{I\cap(-\infty,t]} (t-x)\, d\mu(x) \geq 0 \text{ and } \int_{I\cap[t,\infty)} (x-t)\, d\mu(x) \geq 0$$

for every $t \in [a, b]$.

See also [4], p. 179, for details.

Two examples of nonpositive Steffensen-Popoviciu measures on $[a, b]$ are

$$\frac{5}{9}\delta_{\frac{3a+b}{4}} - \frac{1}{9}\delta_{\frac{a+b}{2}} + \frac{5}{9}\delta_{\frac{a+3b}{4}} \quad \text{and} \quad \left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right) dx.$$

The Steffensen-Popoviciu measures provide the natural framework for the Jensen-Steffensen inequality:

Theorem 2. (*See* [4], *Theorem* 4.2.1, *pp.* 184-185). *Suppose that μ is a Steffensen-Popoviciu measure on an interval I. Then for every continuous convex function f on I,*

$$f(b_\mu) \leq \frac{1}{\mu(I)} \int_I f(x)\, d\mu(x).$$

Here $b_\mu = \frac{1}{\mu(I)} \int_I x\, d\mu(x)$ represents the barycenter of μ.

Proof: As is well known, every continuous convex function is the lower envelope of all affine functions it dominates. See [4], Theorem 1.5.2, p. 31. Technically, this means that

$$f(x) = \sup \{h(x) : h \leq f,\ h \text{ affine}\}$$

for every $x \in I$. Therefore

$$\begin{aligned} f(b_\mu) &= \sup \{h(b_\mu) : h \leq f,\ h \text{ affine}\} \\ &= \sup \left\{ \frac{1}{\mu(I)} \int_I h(x)\, d\mu(x) : h \leq f,\ h \text{ affine} \right\} \\ &\leq \frac{1}{\mu(I)} \int_I f(x)\, d\mu(x). \end{aligned}$$

□

A signed measure is not monotonic. However, if f and h are two continuous real functions defined on an interval I such that f is convex, h is affine and $f \geq h$, then

$$\int_I f(x)\, d\mu(x) \geq \int_I h(x)\, d\mu(x)$$

for every Steffensen-Popoviciu measure μ. What happens when $h \geq f$? Simple examples show that an inequality of the form

$$\int_I h(x)\, d\mu(x) \geq \int_I f(x)\, d\mu(x)$$

may fail for certain Steffensen-Popoviciu measures μ. Of course, it works for all positive measures on I. Does it work outside the framework of positive measures? We will answer this question by considering an appropriate abstract setting.

Definition 2. A real Borel measure μ on an interval I is said to be a *dual Steffensen-Popoviciu measure* if $\mu(I) > 0$ and

$$\int_I f(x)\, d\mu(x) \geq 0 \quad \text{for every nonnegative concave function} \quad f : I \to \mathbb{R}.$$

Two examples of dual Steffensen-Popoviciu measures on the interval $[a, b]$ are

$$-\delta_a + \delta_{\frac{3a+b}{2}} + \delta_{\frac{a+b}{2}} + \delta_{\frac{a+3b}{2}} - \delta_b \text{ and } \left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right) dx.$$

The case of the discrete measure is straightforward. See [4], Corollary 1.4.3, p. 28. Notice that this is an example of a dual Steffensen-Popoviciu measure that is not a Steffensen-Popoviciu measure. The fact that $\left(\left(\frac{2x-a-b}{b-a}\right)^2 - \frac{1}{6}\right) dx$ is a dual Steffensen-Popoviciu measure is a consequence of the following result applied to

$$\varphi(x) = (x-a)(x-b)\left(x - \frac{a+b}{2}\right)^2.$$

Theorem 3. *Suppose that $\varphi \in C^2([a, b])$ is a function such that the following two conditions are fulfilled:*

$(DSP1)$ $\varphi(a) = \varphi(b) = 0$ and $\varphi(x) \leq 0$ for every $x \in [a, b]$;

$(DSP2)$ $\varphi'(a) \leq 0 \leq \varphi'(b)$ and $\varphi'(a) < \varphi'(b)$.

Then $\varphi''(x)dx$ is a dual Steffensen-Popoviciu measure on the interval $[a, b]$.

Proof: Clearly, the measure $d\mu(x) = \varphi''(x)dx$ verifies the condition $\int_a^b d\mu(x) > 0$. In order to prove that $\int_a^b f(x)\, d\mu(x) \geq 0$ for every nonnegative concave function $f : [a, b] \to \mathbb{R}$ we may restrict ourselves to the case where f is also continuous. This can be done by modifying the values of f at the endpoints if necessary. See [4], Proposition 1.3.4, p. 22.

Since

$$f = s + (f - s),$$

where s is the secant line joining the endpoints of the graph of f, we are led to consider separately the following two special cases:

Case 1: $\int_a^b s(x)\, d\mu(x) \geq 0$ for every nonnegative affine function s;

and

Case 2: $\int_a^b g(x)\, d\mu(x) \geq 0$ for every nonnegative continuous concave function g that vanishes at the endpoints.

It is easy to see that Case 1 is equivalent to

$$\int_a^b (x-a)\, d\mu(x) \geq 0 \text{ and } \int_a^b (b-x)\, d\mu(x) \geq 0,$$

and this is covered by our hypotheses.

Via an approximation argument we can reduce Case 2 to the situation when g is a piecewise linear nonnegative concave function associated to a division $a = x_0 < x_1 < ... < x_n = b$ such that $g(a) = g(b) = 0$.

Under these circumstances the fact that Case 2 is covered by (DSP1) can be established by a direct computation:

$$\begin{aligned}\int_a^b g(x)d\mu(x) = \int_a^b g(x)\varphi''(x)dx = g(x)\varphi'(x)|_a^b - \int_a^b g'(x)\varphi'(x)dx\\ = -\int_a^b g'(x)\varphi'(x)dx\\ = \sum_{k=0}^{n-1} -g'(x)\varphi(x)|_{x_k}^{x_{k+1}}\\ + \int_a^b g''(x)\varphi(x)dx\\ = \sum_{k=1}^{n-1} \left[g'_+(x_k) - g'_-(x_k)\right]\varphi(x_k) \geq 0.\end{aligned}$$

The last sum is nonnegative since $g'_+(x_k) - g'_-(x_k) \leq 0$ for a concave function.

□

The Steffensen-Popoviciu measures and the dual Steffensen-Popoviciu measures allow us to extend Theorem 1 beyond the framework of positive measures. Indeed, an inspection of the argument of Theorem 1 easily yields the following much more general result:

Theorem 4. *The statement of Theorem 1 remains valid if μ is a real Borel measure on I and A, B, C are three nonoverlapping subintervals of I such that the restriction of μ to each of the intervals A and C is a Steffensen-Popoviciu measure and the restriction of μ to B is a dual Steffensen-Popoviciu measure.*

An application illustrating Theorem 4 is as follows. Consider two positive numbers a and b. Since

$$\begin{aligned}\frac{b}{6a}\int_{-a}^{a} dx = \frac{1}{3}b\\ = \int_{-a-b}^{-a} \left(\left(\frac{2x+2a+b}{b}\right)^2 - \frac{1}{6}\right) dx\\ + \int_a^{a+b} \left(\left(\frac{2x-2a-b}{b}\right)^2 - \frac{1}{6}\right) dx\end{aligned}$$

and

$$\frac{b}{6a}\int_{-a}^{a} xdx = 0$$
$$= \int_{-a-b}^{-a} x\left(\left(\frac{2x+2a+b}{b}\right)^2 - \frac{1}{6}\right)dx$$
$$+ \int_{a}^{a+b} x\left(\left(\frac{2x-2a-b}{b}\right)^2 - \frac{1}{6}\right)dx,$$

we infer from Theorem 4 that

$$\frac{b}{6a}\int_{-a}^{a} f(x)dx \leq \int_{-a-b}^{-a}\left(\left(\frac{2x+2a+b}{b}\right)^2 - \frac{1}{6}\right)f(x)dx$$
$$+ \int_{a}^{a+b}\left(\left(\frac{2x-2a-b}{b}\right)^2 - \frac{1}{6}\right)f(x)dx \tag{2}$$

for every convex function $f : [-a-b, a+b] \to \mathbb{R}$. In particular, for $a = b = 1$, we get

$$\frac{1}{6}\int_{-2}^{2} f(x)dx \leq \int_{-2}^{-1} (\ 2x+3)^2 f(x)dx + \int_{1}^{2} (\ 2x-3)^2 f(x)dx.$$

As is well known, the large values of a convex function are concentrated near the endpoints. The inequalities (2) reveal that the main feature of Theorem 4 is to provide a quantitative measure of this phenomenon.

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